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The Statistical Prediction of Solutions of Dynamic Equations

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The Statistical Prediction of Solutions of Dynamic Equations*

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Abstract

The current failure of linear regression methods to yield nearly perfect weather forecasts may result because weather is basically unpredictable from currently available initial data, or because linear methods are inadequate. The latter possibility has been tested by generating a series of "weather maps" by numerical integration of a set of nonlinear differential equations, and then attempting to repredict these maps by linear regression.

A two-layer baroclinic model, in which the flow pattern in each layer is expressed by six terms in a double Fourier series, has been integrated on a small electronic computer. The model contains frictional damping, and thermal forcing which is constant with time but variable with latitude and longitude.

The solutions exhibit irregular fluctuations. A spectral analysis reveals a continuous spectrum with most of the variance in a band covering periods from 20 to 60 days, and almost no variance in periods shorter than 4 days. Covariances based upon 4½ years of "data" indicate that linear regression methods give excellent forecasts one day in advance, but only mediocre forecasts more than three days in advance.

In this talk I should like to describe an experiment which has been performed by the Statistical Forecasting Project at M.I.T. For several years our project has been investigating the feasibility of forecasting sea-level pressures by linear regression methods. Our best formulas have given reductions of variance of about 65 percent, for 24-hour forecasting in the North American region. These predictions, although much better than guesses, are still far from perfect, and many of the prognostic pressure maps would not be considered satisfactory by the average forecaster. It has been our feeling that most of the dynamical forecasting models also fail to give satisfactory forecasts of the sea-level pressure.

Since both the statistical and the dynamical methods seem to fail, some meteorologists have suggested that the sea-level pressure field simply cannot be predicted, at least not from observational data which are available at present. Other meteorologists, and particularly those who have favored dynamical forecasting methods, have suggested that it is the linear regression methods which are inadequate, and that suitable forecasts will eventually be made with more refined dynamic models.

Those who have favored linear regression methods can point to a theorem of WIENER [8], who has shown that if a statistically stationary system is deterministic, so that it may be predicted exactly from its own present and past by some formula, it may also be predicted exactly from its own present and past by some linear formula. However, the exact formula may involve an infinite amount

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of past data, and many terms may be required for a good approximation. From the meteorologist's point of view, the important question is whether there is a nearly perfect linear formula involving data from the recent past only.

In the Statistical Forecasting Project, we have proposed to investigate this question by obtaining a numerical solution of a set of deterministic equations, and then investigating the predictability of this solution by linear regression methods. For this purpose our deterministic equations should be nonlinear, since the linear prediction of the solution of a linear equation is trivial. We must exclude equations whose solutions become infinite, since the solutions must be statistically stationary. We should also exclude equations whose solutions do not vary with time, or vary in too regular a manner, since these solutions are very easily predicted by linear methods. Finally, the statistics of the solutions ought to be independent of the chosen initial conditions, so that we should exclude equations which are known to possess certain integral invariants, such as an energy integral.

Since we are ultimately interested in the weather, which appears to constitute a time series whose prediction is not trivial, it seems logical to choose for our deterministic equations one of the simpler models used in numerical weather prediction. Such an equation has the added advantage that the nonlinearity should be of the type encountered in weather forecasting, and may offer the same sort of obstacles to linear prediction.

Accordingly, for our deterministic equations, we have chosen the geostrophic form of the two-layer baroclinic model recently described by the writer [6]. We have appended linear terms, representing heating, proportional to the difference between the existing temperature field and a standard temperature field; skin friction, proportional to the flow in the lower layer; and friction at the surface separating the layers, proportional to the vertical shear. Once the numerical values of the three factors of proportionality, which we have denoted by h, h', h'' respectively, have been chosen, it is to be hoped that the statistics of the solution will be determined by the standard temperature field, rather than by the chosen initial conditions.

We have further simplified the equations by suppressing all variations of static stability, so that the model is reduced to one of the more conventional two-layer models. The system is then completely described by the flow in each layer, and is conveniently expressed by half the sum, ψ, and half the difference, τ, of the stream functions for the two layers. The field of τ is identified with the temperature field through the thermal wind equation. We have denoted the corresponding standard temperature field, toward which heating tends to drive the existing temperature field, by τ^*.

We have still further simplified the system by replacing the spherical surface of the earth by an infinite strip, bounded by the lines $y=0$ and $y=\pi/l$ and letting the Coriolis parameter f be constant. Over such a region the functions $\cos mly$, $\sqrt{2}\sin mly \cos nkx$, $\sqrt{2}\sin mly \sin nkx$, for integral values of m and n, form a set of orthogonal functions with equal mean squares. We have expanded ψ and τ into series of such functions, and have then omitted reference to all but six of these functions—those for which $m=1$ or 2 and $n=1$, in a manner recently described by the writer [5]. Thus

$$\psi=\psi_A \cos ly+\psi_C \cos 2ly$$
$$+\sqrt{2}(\psi_K \sin ly+\psi_M \sin 2ly)\cos kx$$
$$+\sqrt{2}(\psi_L \sin ly+\psi_N \sin 2ly)\sin kx\,, \quad (1)$$

with a similar expansion for τ.

Upon substituting the expressions for ψ and τ into the equations of the model, we obtain 12 ordinary differential equations in the 12 variables

$\psi_A, \psi_K, \psi_L, \psi_C, \psi_M, \psi_N, \tau_A, \tau_K, \tau_L, \tau_C, \tau_M, \tau_N$. If these variables are also denoted by $X_1, X_2, \ldots, X_{12}$, respectively, the equations may be written

$$\frac{dX_i}{dt}=\sum_{m,n} a_{imn}X_mX_n+\sum_m b_{im}X_m+c_i\,. \qquad (2)$$

In equations (2), the quadratic terms $a_{imn}X_mX_n$ represent the advective effects. The linear terms $b_{im}X_m$ represent the effects of friction, and part of the effects of heating. The constant terms c_i, which are different from zero only in the last six equations, depend upon the standard temperature field τ^* toward which the temperature field is being driven by the effects of heating. Equations (2) closely resemble the set which has been integrated numerically by BRYAN [2].

To obtain numerical values, we have chosen one half of the earth's radius, or $10000/\pi$ kilometers, as the unit of distance. We have let the width W of the strip and the fundamental wave length in the eastward direction each equal 10,000 kilometers, so that $l=1$ and $k=2$. We have chosen the unit of time to be three hours, approximately the reciprocal of the Coriolis parameter in middle latitudes.

Finally we have chosen the values 1/16, 1/16, and 1/64 for h, h', and h'', respectively, and have chosen the value 0.16 for the dimensionless static stability parameter

$$\sigma_0=\frac{1}{16}\pi^2W^{-2}f^{-2}c_p\Delta\theta\,,$$

which enters the two-layer model, $\Delta\theta$ being the diffenence between the mean potential temperatures of the two layers.

By omitting the higher-order terms in the Fourier series, we have excluded all but the largest scales of motion. The resemblance between such a simplified system and the atmosphere is indicated by the two maps in Fig. 1, which show the fields of ψ for two different sets of values of ψ_A, ψ_K, ψ_L, ψ_C, ψ_M and ψ_N. The second map follows the first by 3 days, according to the integration of a finite difference form of the governing equations, which will presently be described.

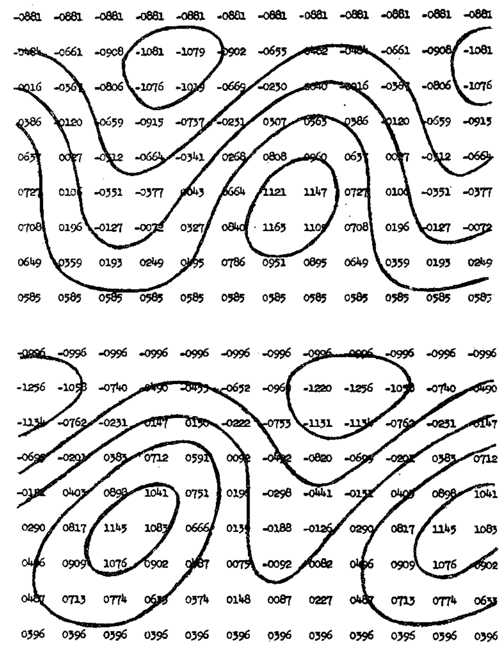

Fig. 1. Fields of ψ over a region covering $1\frac{1}{2}$ wave lengths.

Upper: $\psi_A=.0733$, $\psi_K=.0348$, $\psi_L=-.0467$, $\psi_C=-.0148$, $\psi_M=-.0110$, $\psi_N=.0159$

Lower: $\psi_A=.0696$, $\psi_K=-.0319$, $\psi_L=.0423$, $\psi_C=-.0300$, $\psi_M=.0229$, $\psi_N=-.0197$

The maps show a succession of subtropical anticyclones and subpolar cyclones. The systems move about one-half wave length eastward during the period, and, in accordance with the NE-SW tilt of the trough lines, and the accompanying northward transport of eastward momentum, the belt of maximum westerly winds is observed to shift northward.

The Statistical Forecasting Project has a small electronic computer, a Royal-McBee LGP-30, with a memory of 4,096 words. This machine can solve the 12 equations by an iterative procedure, at the rate of about 10 seconds per time step. Because we have retained features of only the largest scale, the solution is computationally stable with 6-hour time steps. Thus we can generate nearly six years of data in 24 hours of machine time.

Our first problem was to determine whether the solutions of these equation were suitable for the statistical tests. The solutions obviously depend upon the nature of the thermal forcing function, which depends upon τ^*. In each run, τ^* was of the form

$$\tau^* = \tau_A^* \cos y + \tau_C^* \cos 2y + \sqrt{2}\,\tau_M^* \sin 2y \cos 2x.$$

The coefficients τ_A^*, τ_C^* and τ_M^* were not allowed to vary with time within a run, although they frequently had different values in different runs.

In the first set of runs, the field of τ^* was simply a multiple of $\cos y$, so that τ_C^* and τ_M^* were zero. With values of τ_A^* near 0.1, the resulting solutions proved to oscillate with two degrees of freedom. The waves progressed from west to east, traveling one wave length in about six days. At the same time, the waves underwent a periodic change in their shape, with a period of about 20 days. This phenomenon is the vacillation which has been obtained experimentally by HIDE [3]. Obviously, the statistical prediction of such a regular phenomenon is trivial.

In the next set of runs, τ^* was still a function of latitude only, but both τ_A^* and τ_C^* were different from zero, with τ_C^* smaller than τ_A^*. In some runs pure vacillation occurred again, but, with $\tau_A^*=0.1$, and τ_C^* between -0.018 and -0.026, the successive vacillation cycles themselves showed a periodic change, so that the system oscillated with at least three degrees of freedom. However, aside from a phase shift in longitude, the initial situation usually recurred within a year, and there was little evidence of the irregular fluctuations which characterize the atmosphere, and which make statistical forecasting so difficult.

Our final modification of the field of τ^* was to add a variation with longitude, by letting τ_M^* also be different from zero. We thereby simulated a distribution of continents and oceans. This change could have the effect of making two weather situations, initially alike except for a longitudinal phase difference, develop quite differently. It should therefore add new degrees of freedom to the oscillations of the system.

In this set of runs, with $\tau_A^*=0.1$, $\tau_C^*=-0.025$, and $\tau_M^*=-0.025$, the solutions showed the irregularity which we had been seeking. In runs totaling more than twenty years, we have not yet found any repetitions of previous conditions.

Fig. 2 presents a graph of the variations of $-\psi_C$ over a period of eight months. This quantity is high when the zonal westerly winds are displaced to the north of their normal position, and low when they are displaced to the south. We may regard it as a form of zonal index.

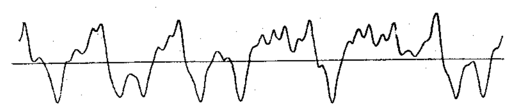

Fig. 2. Graph of $-\psi_C$ against time for a particular eight-month period. Horizontal line is zero line.

We see that there are large fluctuations with irregular periods of a month or two, and, at times of high zonal index, some minor fluctuations with a period of about six days. The latter period is the cyclone period, and we may regard the former oscillations as constituting the index cycle.

Perhaps the most significant feature of this graph is its randomness. The different portions of the curve have certain preferred shapes, but these shapes do not follow one another in any regular order. To examine the randomness further, we

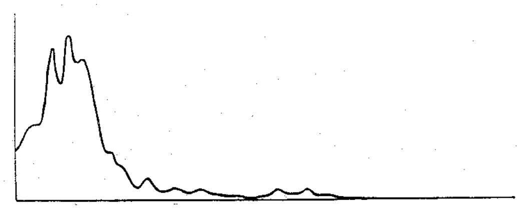

Fig. 3. Power spectrum of ψ_C for 2800-day period, computed from autocorrelations up to 200 days' lag.

have obtained a power spectrum of the variable ψ_G, by obtaining autocorrelations up to 200 days' lag from a record of 2,800 days, and then taking the Fourier transform of the autocorrelation function, in the manner set forth by BLACKMAN and TUKEY [1]. The spectrum is shown in Fig. 3.

What we observe is a continuous spectrum rather than a line spectrum. Conceivably the true spectrum is a line spectrum, but, if this is so, there are so many lines that they cannot be resolved by sampling the spectrum at 200 points. From the point of view of practical prediction, the spectrum is continuous.

The most conspicuous feature is the broad band covering periods of 20 to 60 days, which contains most of the total variance. A small but significant amount of variance is also contained in a band covering periods of 6 to 7 days. There is an almost complete absence of variance in periods shorter than 4 days.

The spectrum thus bears considerable resemblance to the spectra of various actual atmospheric variables which have frequently been presented. It differs principally in that the true atmospheric spectra do not show such a nearly complete absence of variance in the higher frequencies. This absence of variance in the spectrum of ψ_G was to be anticipated from the integration procedure, which used six-hour time steps. If the solutions of the differential equations has possessed significant oscillations in periods of a day or two, the finite-difference equations would not have been computationally stable with six-hour time steps.

It would then appear that any numerical solution obtained by a computationally stable stepwise integration procedure will be nearly devoid of variance in periods equal to a few time steps or less, and in this respect will fail to resemble the weather. Time series behaving more like the atmosphere can be constructed by letting the period between successive observations be reasonably long compared to the time step used in the integration.

The form of the power spectrum is closely related to the linear predictability. A result obtained independently by KOLMOGOROFF [4] and WIENER [7] states that in predicting a simple time series from its own past, one time step in advance, the ratio of the variance of the error in prediction to the variance of the original series equals the ratio of the geometric mean of the spectrum to the arithmetic mean. If a series is to be highly predictable from its own past, then, the extreme values of the spectrum must differ by several orders of magnitude.

Clearly, then, the time series of values of ψ_G at one-day intervals is highly predictable from its own past, since about one half of the spectrum lies virtually on the zero-line. Its value cannot be exactly zero, but it is as small as the computational error. If, however, we form a time series from values of ψ_G at two-day intervals, the cut-off frequency in the spectrum is one cycle per four days, and no portion of the spectrum is extremely close to zero. There then remains a considerable error in prediction. In this case the reduction of variance is found to be 85 per cent.

The computations based upon a single spectrum pertain to the reduction of variance in predicting a single variable from its own past. The nonlinear equations generating each variable involve all twelve variables as predictors. In comparing linear and nonlinear prediction, then, we should investigate the predictability of each variable, using present and past values of all the variables as predictors.

In this problem also the interval between successive observations of the predictors is important. Since the integration is performed in six-hour time steps, it would be possible to use observations of the predictors separated by as little as six hours. However, our experience with the spectrum of ψ_G suggests that nearly perfect prediction will be obtained unless the observations are separated by at least two days.

Accordingly, from 4½ consecutive years of observations at one-day intervals, we have computed the covariance of each predictor with itself and with each other predictor, at lags ranging up to 12 days. From these we have computed reductions of variance in predicting one, two, and four days in advance, which are presented in Table 1.

Table 1. Reduction of total variance in linear prediction of twelve variables, with indicated times of predictors and predictands.

Predictor days	Predictand day	Reduction of variance
0	+1	.972
-1, 0	+1	.997
-2, -1, 0	+1	.999
0	+2	.912
-1, 0	+2	.977
-2, -1, 0	+2	.990
0	+2	.912
-2, 0	+2	.964
-4, -2, 0	+2	.976
0	+4	.715
-2, 0	+4	.841
-4, -2, 0	+4	.883
0	+4	.715
-4, 0	+4	.811
-8, -4, 0	+4	.854

The results show that, as anticipated, prediction one day in advance is nearly perfect when the predictors occur at one-day intervals. Predictions two days ahead, while very good, are not perfect, and predictions four days in advance fall far short of perfection. It should also be noted that these reductions of variance have been obtained from a single sample, so that the true reductions of variance for the population are probably somewhat lower.

Since the computed results pertain to a particular numerical model, and not to the real atmosphere, with its numerous complications not included in the model, it is not possible to draw definite conclusions concerning the atmosphere itself. However, the work demonstrates the existence of deterministic systems, governed by equations whose nonlinearity resembles the nonlinearity of the atmosphere, which are not perfectly nor almost perfectly predictable by simple and simply determined linear formulas, if the period between successive observations is greater than half of the shortest significant period of oscillation. Since the atmosphere possesses significant oscillations with periods of fractions of a day, as well as random errors of observation which add a certain amount of white noise to the spectrum, it is suggested that in predicting the atmosphere as much as 24 hours ahead, better results should eventually be obtained by some nonlinear statistical procedure, or by the methods of dynamical weather prediction, than can be obtained by purely linear prediction.

Acknowledgement

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DISCUSSION

Bolin: Did you change the initial condition just slightly and see how much different results were in the forecasting in this way?

A: As a matter of fact, we tried out that once with the same equation to see what could happen. We changed one of the 12 variables by a factor of a small fraction of 1%, a change which would be considered to be smaller than observational error. We found that this error grew and continued to grow at a slow exponential rate. After 1 or 2 months, it is still pretty small so the map looked about right but it is comparable with the observational error. But after 6 months there is no resembrance at all between the 2 maps, which implied that at least for this particular set of equations there is a limit to how far you can forecast. Thus by these dynamic methods if you assume you have any observational error whatever to begin with, eventually the error predominates. Each of the series has the same statistics.
But they were simply different series after about 6 months.

Mount: What loss in prediction accuracy can be expected on independent data using your theoretical formula?

A: Just a few percentage units. I have not computed it, however.

Platzman: Why do you assume that sensitivity of the prediction to a slight variation in the initial data is in someway connected with the small numbers of parameters in the model?

A: I cannot really give an answer to that question. Yet my feeling is that perhaps the small number of parameters over-emphasizes the sensitivity rather than under-emphasizes. I do know if you have very small numbers you cannot get anything but motion without change of shape or oscillation. In that case it is not at all sensitive in that sense. You have a small error which will not grow, but stay about the same. When you have as many variables as this, apparently it is very sensitive. If we brought, say, 50 or 1000 variables or something like that I just cannot say. It will be pure speculation.

Estoque: Did you try to make exactly the same run, but just changing Δt, say, 1/2 or 3 hours instead of 6 hours?

A: We did try. Even changing 6 hours to 12 hours, we would have essentially the same type of solution. They would not agree exactly, that is, the period in one case might change from 30 days to 33 days or something like that. It would be the same general type of thing but there would be some differences. I suspect that in this case that if we cut it down to 3 hours or 1 hour or something like that, there would be significant differences in the solution after a certain amount of time.